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On new Galilei-invariant equations in two-dimensional spacetime

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Abstract. We study realizations of Galilei groups acting as transformation Lie groups in the space of two independent variables and one dependent variable. Classification of realizations of the Lie algebras $AG_1(1,1)$, $AG_2(1,1)$, $AG_3(1,1)$, $A\tilde{G}_1(1,1)$, $A\tilde{G}_2(1,1)$ and $A\tilde{G}_3(1,1)$ within the class of Lie vector fields is carried out. Utilizing the classification results we have constructed the full sets of second-order scalar differential equations in two-dimensional spacetime invariant under the Lie algebras $AG_1(1,1)$ and $A\tilde{G}_1(1,1)$.

1. Introduction (basic notations and definitions)

A usual restriction imposed on the choice of a mathematical model (differential equation(s)) in non-relativistic physics is that it has to obey the Galilei relativity principle. This means that on the set of solutions of the differential equation in question some realization of the Galilei group is to be realized. Consequently, to be able to present a complete set of all possible equations satisfying the Galilei relativity principle one has to solve an intermediate problem of describing all inequivalent (in some sense) realizations of the Galilei group by Lie vector fields [1–4].

In this paper we turn to non-relativistic theories, namely, Galilei invariant ones in $1+1$ dimensions and construct the most general equation

$$F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0 \tag{1}$$

invariant under the Galilei group. In equations (1) $u(t, x)$ is a real-valued scalar function, subscripts denote partial derivatives, and F is a smooth real-valued function of the indicated variables.

The set of equations (1) contains, in particular, such linear and nonlinear Galilei invariant equations as the heat, the Burger's and the modified Burger's equations [1, 2, 5]. 'Invariance' is always meant in the 'strong' sense, i.e. if a equation (1) is invariant under some one-parameter group $\exp \lambda Q$, then this equation is annihilated by the second-order prolongation $pr^{(2)}Q$ of Q everywhere, not only on the solution set of (1).

The first step in the derivation of invariant equations is to realize the Lie algebra of the assumed symmetry group in terms of vector fields on the space $X \otimes U$ of independent and dependent variables. In our case, X is the two-dimensional space with coordinates t, x and U is the space of real scalar functions $u(t, x)$. The vector fields will all have the form

$$Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u \tag{2}$$

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where τ, ξ, η are some sufficiently smooth real-valued functions on the space $X \otimes U$.

Let $AL = \langle Q_1, \ldots, Q_N \rangle$ be a Lie algebra and its basis operators Q_i satisfy commutation relations

$$[Q_k, Q_m] = C^n_{km} Q_n \tag{3}$$

where C^n_{km} are real-valued constants (structure constants), $k, m, n = 1, 2, \ldots, N$.

We say that operators $Q_i (i = 1, \ldots, N)$ of the form (2) realize a representation (realization by Lie vector fields) of the Lie algebra AL if:

- they are linearly independent,
- they satisfy the commutation relations (3).

It is straightforward to verify that relations (3) are not altered by an arbitrary invertible transformation of the independent and dependent variables

$$t' = h(t, x, u) \qquad x' = g(t, x, u) \qquad u' = f(t, x, u) \tag{4}$$

where h, g, f are sufficiently smooth functions. Invertible transformations (4) form a group (called a diffeomorphism group) which establishes an equivalence relation on the set of all possible realizations of the Lie algebra AL. Two realizations of the Lie algebra AL are called equivalent if the corresponding basis operators can be transformed into one another by a change of variables (4).

In what follows we make use of the following classification of Lie algebras of Galilei groups (called in the following the Galilei algebras) [6].

The Lie algebra $AG_1(1, 1) = \langle T, P, G \rangle$ is called the *classical* Galilei algebra if its basis operators satisfy following commutation relations:

$$[T, P] = 0 \qquad [T, G] = -P \tag{5}$$

$$[P, G] = 0. \tag{6}$$

The Lie algebra $AG_2(1, 1) = \langle T, P, G, D \rangle$ is called the *special* Galilei algebra if its basis operators satisfy the commutation relations (5), (6) and the relations

$$[D, P] = -P \qquad [D, G] = G \qquad [D, T] = -2T. \tag{7}$$

The Lie algebra $AG_3(1, 1) = \langle T, P, G, D, S \rangle$ is called the *complete* Galilei algebra if its basis operators satisfy commutation relations (5)–(7) and relations

$$[S, G] = 0 \qquad [S, P] = G \qquad [T, S] = D \qquad [D, S] = 2S. \tag{8}$$

Let an operator M satisfy the following commutation relations:

$$[M, T] = [M, P] = [M, G] = [M, D] = [M, S] = 0 \tag{9}$$

$$[G, P] = M. \tag{10}$$

The Lie algebras $A\tilde{G}_1(1, 1) = \langle T, P, M, G \rangle$, $A\tilde{G}_2(1, 1) = \langle T, P, M, G, D \rangle$, $A\tilde{G}_3(1, 1) = \langle T, P, M, G, D, S \rangle$ are called the *extended classical* Galilei algebra, the *extended special* Galilei algebra, the *extended complete* Galilei algebra (the Schrödinger algebra) if their basis operators satisfy commutation relations (5), (7)–(10).

Note that the Burger's equation is invariant under the complete Galilei algebra. Furthermore, the heat and the modified Burger's equations are invariant under the extended complete Galilei algebras [1, 2, 5]. Rideau and Winternitz [7] have obtained a complete description of the realizations of the extended classical Galilei algebra and its generalizations provided the generators of the time T, space P and phase M translations can be simultaneously rectified to become

$$T = \partial_t \qquad P = \partial_x \qquad M = \partial_\phi. \tag{11}$$

(Note that they considered the case of two dependent and two independent variables.) The results were used to obtain the general forms of the second order evolution equations

$$\psi_t + F(t, x, \psi, \psi^*, \psi_x, \psi_x^*, \psi_{xx}, \psi_{xx}^*) = 0$$

invariant under the $A\tilde{G}_1(1,1)$, $A\tilde{G}_2(1,1)$ and $A\tilde{G}_3(1,1)$ algebras.

As noted by Zhdanov and Fushchych [8, 9] this is an additional restriction that does not follow from the definition of realization of a Lie algebra and thus leads to losing some classes of Galilei-invariant equations. They have completed the classification of realizations of the extended classical, extended special and extended complete Galilei algebras in two independent and two dependent variables. This approach has also been applied to classify all scalar equations of the form (1) invariant under the Poincaré, extended Poincaré and conformal algebras [10, 11].

The above papers by Rideau and Winternitz and by Zhdanov and Fushchych consider the case of the extended classical Galilei algebra, which means that $M \neq 0$. In this paper we study both the case $M \neq 0$ and the case of vanishing M, which means that we also take into consideration realizations of the classical Galilei algebra. What is more, we do not require that the operators T, P, M are reducible to the form (11). The last remark is that we construct the general Galilei-invariant equations (not only evolution type ones) for one function of two variables.

This paper is organized as follows. In sections 2 and 3 we obtain all realizations of the Lie algebras $AG_1(1,1)$, $AG_2(1,1)$, $AG_3(1,1)$, $A\tilde{G}_1(1,1)$, $A\tilde{G}_2(1,1)$, $A\tilde{G}_3(1,1)$ by Lie vector fields (LVFs) (2). In section 4 we obtain second-order differential invariants of the prolonged group action for the above groups which yields the Galilei-invariant equations (1).

2. Realizations of the classical, special and complete Galilei algebras

Before formulating the principal assertion we give an auxiliary lemma.

Lemma 1. Let T, P be mutually commuting linearly independent operators of the form (2). Then there exists transformation (4) reducing these operators to one of the forms below

$$T = \partial_t \qquad P = -\partial_x \tag{12}$$

$$T = \partial_t \qquad P = -x\partial_t. \tag{13}$$

Proof. Let A be a 2×3 matrix whose entries are coefficients of the operators T, P.

Case 1: rank $A = 2$. It is a common knowledge that any non-zero operator Q of the form (2) having smooth coefficients can be transformed by the change of variables (4) to become $Q' = \partial_{t'}$, (see, e.g. [1]). Consequently, without loosing generality, we can suppose that the relation $T = \partial_t$ holds (hereafter we skip the primes). As the operator P commutes with T, its coefficients do not depend on t, i.e.

$$P = \tau(x,u)\partial_t + \xi(x,u)\partial_x + \eta(x,u)\partial_u.$$

By assumption, one of the coefficients τ, ξ, η is not equal to zero. Without loss of generality, we can suppose that $\xi \neq 0$ (if this is not the case, we make a change $x \to u$, $u \to x$). Performing the transformation

$$t' = t + f(x,u) \qquad x' = g(x,u) \qquad u' = h(x,u)$$

where the functions f, g, h are solutions of partial differential equations (PDEs)

$$Pf + \tau = 0 \qquad P\xi = -1 \qquad Ph = 0$$

we reduce the operators T, P to become $T = \partial_t, P = -\partial_x$.

Case 2: rank $A = 1$. If me make transformation (3) reducing the operator T to the form $T = \partial_t$, then the operator P becomes $P = \tau(x, u)\partial_t$ (the function τ does not depend on t because T and P commute). As $\tau \neq$ constant (otherwise the operators T and P are linearly dependent), making the change of variables

$$t' = t \qquad x' = -\tau(x, u) \qquad u' = h(x, u) \qquad \frac{D(\tau, h)}{D(x, u)} \neq 0$$

transforms the operators T, P to be $T = \partial_t, P = -x\partial_t$. □

Theorem 1. Inequivalent realizations of the classical Galilei algebra by LVFs (2) are exhausted by the following realizations:

$$\begin{aligned}
&(1)\ T = \partial_t \qquad P = -\partial_x \qquad G = t\partial_x \\
&(2)\ T = \partial_t \qquad P = -\partial_x \qquad G = u\partial_t + t\partial_x \\
&(3)\ T = \partial_t \qquad P = -\partial_x \qquad G = t\partial_x + \partial_u \\
&(4)\ T = \partial_t \qquad P = -x\partial_t \qquad G = xt\partial_t + x^2\partial_x.
\end{aligned} \tag{14}$$

Proof. To prove the theorem it suffices to solve the commutation relations (5), (6) for the basis operators T, P, G of the classical Galilei algebra within the class of LVFs (2) up to the action of the diffeomorphism group (4). All inequivalent realizations of the two-dimensional commutative algebra having the basis operators T, P are given by formulae (12), (13). What is left is to solve the commutation relations for the generator of the Galilei transformations $G = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$

$$[T, G] = -P \qquad [P, G] = 0 \tag{15}$$

for each set of operators T, P listed in (12), (13).

Let operators T, P have the form (12). From the commutation relations (15) we find that G must have the form

$$G = \tau(u)\partial_t + (t + \xi(u))\partial_x + \eta(u)\partial_u. \tag{16}$$

If in (16) $\eta = 0$, then performing the transformation

$$t' = t + \xi \qquad x' = x \qquad u' = u$$

we reduce the operators T, P, G to the form

$$T = \partial_{t'} \qquad P = -\partial_{x'} \qquad G = \tilde{\tau}(u')\partial_{t'} + t'\partial_{x'}. \tag{17}$$

If in (17) $\tilde{\tau} =$ constant, then operators T, P, G realize a representation of the algebra $AG_1(1, 1)$ that is equivalent to the first realization from (14). Next, if in (17) $\tilde{\tau}_{u'} \neq 0$, then performing the transformation

$$t'' = t' \qquad x'' = x' \qquad y'' = \tilde{\tau}(u')$$

we reduce the operators T, P, G to the form 2 from (14).

If in (16) $\eta \neq 0$, then performing the transformation

$$t' = t + h(u) \qquad x' = x + g(u) \qquad u' = f(u) \qquad f_u \neq 0$$

where the functions h, g, f are solutions of equations

$$Gh + \tau = 0 \qquad Gg + \xi = h \qquad Gf = 1$$

we reduce the operators T, P, G to the form 3 from (14).

Finally, let the operators T, P be given by (13). From the commutation relations (15) we find that G must have the form

$$G = [tx + \tau(x,u)]\partial_x + x^2\partial_x + \eta(x,u)\partial_u.$$

Performing the transformation

$$t' = t + h(x,u) \qquad x' = x \qquad u' = f(x,u)$$

where the functions h, f are solutions of PDEs

$$\tau + x^2 h_x + \eta h_u = xh \qquad x^2 f_x + \eta f_u = 0$$

we reduce the operators T, P, G to become $T = \partial_t,\ P = -x\partial_t,\ G = xt\partial_t + x^2\partial_x$. □

Corollary 1. Inequivalent realizations of the special Galilei algebra by LVFs (2) are exhausted by the following realizations:

$$(1)\ T = \partial_t \qquad P = -\partial_x \qquad G = t\partial_x$$
$$D = 2t\partial_t + x\partial_x + \epsilon u\partial_u \qquad \epsilon = 0, 1 \tag{18}$$
$$(2)\ T = \partial_t \qquad P = -\partial_x \qquad G = t\partial_x + \partial_u$$
$$D = 2t\partial_t + x\partial_x + (\lambda - u)\partial_u \qquad \lambda \in R \tag{19}$$
$$(3)\ T = \partial_t \qquad P = -x\partial_t \qquad G = xt\partial_x + x^2\partial_x$$
$$D = 2t\partial_t + x\partial_x + \epsilon u\partial_u \qquad \epsilon = 0, 1 \tag{20}$$
$$(4)\ T = \partial_t \qquad P = -\partial_x \qquad G = u\partial_t + t\partial_x$$
$$D = 2t\partial_t + x\partial_x + 3u\partial_u. \tag{21}$$

Corollary 2. Inequivalent realizations of the complete Galilei algebra by LVFs (2) are exhausted by the following realizations:

(1) $T,\ P,\ G,\ D$ have the form (18), where $\varepsilon = 0$

$S = t^2\partial_t + tx\partial_x$

(2) T, P, G, D have the form (18), where $\epsilon = 1$

$S = t^2\partial_t + (tx + \epsilon_1 u^3)\partial_x + u(t + \lambda u^2)\partial_u$

where $\epsilon_1 = \pm 1, \lambda \in R$, or $\epsilon_1 = 0, \lambda = 0, \pm 1$

(3) T, P, G, D have the form (19)

$S = t^2\partial_t + tx\partial_x + (x + (\lambda - u)t)\partial_u, \lambda \in R$

(4) T, P, G, D have the form (20), where $\epsilon = 0$

$S = t^2\partial_t + xt\partial_x$.

Proof of corollaries 1, 2 is analogous to that of lemma 1 and theorem 1. It should be noted that the realization (21) is not extendable to a realization of the complete Galilei algebra.

3. Realizations of the extended Galilei algebras

We start by constructing inequivalent realizations the operators T, P, M.

Lemma 2. Let T, P, M be mutually commuting linearly independent operators of the form (2). Then there exists transformations (4) reducing these operators to one of the forms below

$$T = \partial_t \qquad P = -\partial_x \qquad M = \partial_u \tag{22}$$

$$T = \partial_t \qquad P = -\partial_x \qquad M = \alpha(u)\partial_t + \beta(u)\partial_x \tag{23}$$

$$T = \partial_t \qquad P = -x\partial_t \qquad M = \gamma(x)\partial_t \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}x} \neq \text{constant} \tag{24}$$

$$T = \partial_t \qquad P = -x\partial_t \qquad M = 2u\partial_t \tag{25}$$

$$T = \partial_t \qquad P = -x\partial_t \qquad M = 2\partial_u. \tag{26}$$

Here $\alpha(u)$, $\beta(u)$ are arbitrary real-valued functions and $\mid \alpha_u \mid + \mid \beta_u \mid \neq 0$.

Proof of lemma 2 is analogous to that of lemma 1 (see also [8, 9]).

The obtained realizations (22)–(25) can be further extended to $A\tilde{G}_1(1, 1)$ by adding an operator G. Once the commutation relations (5), (9), (10) are satisfied, further transformations respecting the form of the realizations of $A\tilde{G}_1(1, 1)$ can be performed. The realization (26) of the operators T, P, M cannot be extended to a realization of the Lie algebra $A\tilde{G}_1(1, 1)$. We present below the final results of our calculations.

Theorem 2. Inequivalent realizations of the extended classical Galilei algebra by LVFs (2) are exhausted by the following realizations:

$$\begin{aligned}
&(1)\ T = \partial_t \qquad P = -\partial_x \qquad M = \partial_u, G = t\partial_x + x\partial_u \\
&(2)\ T = \partial_t \qquad P = -\partial_x \qquad M = \varphi\partial_t + u\partial_x \\
&G = x\varphi\partial_t + (t + xu)\partial_x + (u^2 + \varphi)\partial_u, \text{ where } \varphi = 0 \text{ or} \\
&\varphi = \varphi(u) \text{ satisfies relation } 2\varphi(C\varphi - 1) = u^2, C \in R \\
&(3)\ T = \partial_t \qquad P = -x\partial_t \qquad M = \gamma(x)\partial_t \\
&G = xt\partial_t + (x^2 - \gamma(x))\partial_x, \text{ where } \gamma = \gamma(x) \left(\frac{\mathrm{d}\gamma}{\mathrm{d}x} \neq 0\right) \\
&\text{satisfies relation } C\gamma^2 + 2\gamma = x^2, C \in R \\
&(4)\ T = \partial_t \qquad P = -x\partial_t \qquad M = 2u\partial_t \\
&G = tx\partial_t + (x^2 - 2u)\partial_x + ux\partial_u.
\end{aligned} \tag{27}$$

Each of the realizations obtained in theorem 2 can be extended to realizations of the extended special Galilei algebra by adding a dilation operator D. Its form is determined by the commutation relations (7), (9) and can be further simplified by transformations (4). The corresponding results are given in corollary 3.

Corollary 3. Inequivalent realizations of the extended special Galilei algebra by LVFs (2) are exhausted by the following realizations:

$$\begin{aligned}
&(1)\ T = \partial_t \qquad P = -\partial_x \qquad M = \partial_u \\
&G = t\partial_x + x\partial_u \qquad D = 2t\partial_t + x\partial_x + \lambda\partial_u, \lambda \in R
\end{aligned} \tag{28}$$

$$\begin{aligned}
&(2)\ T = \partial_t \qquad P = -\partial_x \qquad M = \varphi\partial_t + u\partial_x \\
&G = x\varphi\partial_t + (t + xu)\partial_x + (u^2 + \varphi)\partial_u \\
&D = 2t\partial_t + x\partial_x + u\partial_u \qquad \varphi = 0 \text{ or } \varphi = -\tfrac{1}{2}u^2
\end{aligned} \tag{29}$$

$$\begin{aligned}
&(3)\ T = \partial_t \qquad P = -x\partial_t \qquad M = \tfrac{1}{2}x^2\partial_t \\
&G = xt\partial_t + \tfrac{1}{2}x^2\partial_x
\end{aligned}$$

$$D = 2t\partial_t + x\partial_x + \epsilon\varepsilon u\partial_u \qquad \epsilon = 0, 1 \tag{30}$$

(4) $T = \partial_t \qquad P = -x\partial_t \qquad M = 2u\partial_t$

$$G = tx\partial_t + (x^2 - 2u)\partial_x + ux\partial_u \qquad D = 2t\partial_t + x\partial_x + 2u\partial_u. \tag{31}$$

Thus obtained realizations (28), (29) of the extended special Galilei algebra can be extended to the realizations of the extended complete Galilei algebra (the Schrödinger algebra) by adding a generator S of projective transformations. Its form is determined by the commutation relations (8), (9) and can be further simplified by transformations (4). Proceeding as above, we find that the realizations (30) and (31) cannot be extended. We present the results of our calculations in corollary 4.

Corollary 4. Inequivalent realizations of the extended complete Galilei algebra by LVFs (2) are exhausted by the following realizations:

(1) T, P, M, G, D have the form (28)

$S = t^2\partial_t + tx\partial_x + (\frac{1}{2}x^2 + \lambda t)\partial_u, \lambda \in R$

(2) T, P, M, G, D have the form (29)

$S = (t^2 - \frac{1}{4}x^2u^2)\partial_t + (tx + \frac{1}{2}x^2u)\partial_x + (t + \frac{1}{2}xu)u\partial_u.$

4. On Galilei-invariant PDEs

In order to obtain invariant equations we use the usual Lie infinitesimal routine [1]. Let $Q_a, a = 1, \ldots, N$, be a basis for the Lie algebra AL of the symmetry group, acting on the space $X \otimes U$. In our case, $X \otimes U$ is the space $\langle t, x, u\rangle$ and all Q_a have the form (2). The equation (1) is invariant under a Lie algebra AL if the function F satisfies system of PDEs

$$pr^{(2)}Q_a \cdot F = 0 \qquad a = 1, \ldots, N \tag{32}$$

where $pr^{(2)}Q_a$ is the second prolongation of the operator Q_a.

Thus all what we have to do is to find the characteristic for the set of equations (32). The characteristics provide us with a set of elementary invariants $J_k(t, x, u, u_\mu, u_{\mu\nu})$, where $(\mu, \nu = t, x)$, $k = 1, \ldots, s$, so that an invariant equation reads as

$$\Psi(J_1, \ldots, J_s) = 0. \tag{33}$$

We look for invariants of the Galilei groups for each realization of the corresponding Lie algebras given in theorems 1 and 2. The number of variables in (1) and (32) is eight. The algebras $AG_1(1, 1)$ and $A\tilde{G}_1(1, 1)$ are solvable and the generic orbits of the corresponding prolonged group action are three- and four-dimensional, respectively. In view of these facts there are five and four functionally independent invariants, respectively.

The result of our calculations can be summarized as follows.

(1) Elementary invariants of the classical Galilei algebra

(1) $J_1 = u \qquad J_2 = u_x \qquad J_3 = u_tu_{xx} - u_xu_{tx}$

$$J_4 = u_{tt}u_{xx} - u_{tx}^2 \qquad J_5 = u_{xx}. \tag{34}$$

(2) $J_1 = u \qquad J_2 = u_x^{-2}(u_t^2 + 2u_x)$

$J_3 = u_x^{-3}[u_t^2u_{xx} - 2u_tu_xu_{tx} + u_x^2u_{tt}] \qquad J_4 = u_{tt}u_{xx} - u_{tx}^2$

$$J_5 = u_tu_x^{-4}[u_x^2u_{tt} - 2u_tu_xu_{tx} + u_t^2u_{xx}] + u_x^{-3}[u_tu_{xx} - u_xu_{tx}]. \tag{35}$$

(3) $J_1 = u_t + uu_x \qquad J_2 = u_x \qquad J_3 = u_tu_{xx} - u_xu_{tx}$

$$J_4 = u_{tt}u_{xx} - u_{tx}^2 \qquad J_5 = u_{xx}. \tag{36}$$

(4) $J_1 = u \qquad J_2 = xu_t \qquad J_3 = x^2 u_{tt}$

$J_4 = x^3[u_t^2 + x(u_t u_{tx} - u_x u_{tt})]$

$$J_5 = x^4[u_t^2 + 2x(u_t u_{tx} - u_x u_{tt}) - x^2(u_{tt}u_{xx} - u_{tx}^2)]. \tag{37}$$

(2) Elementary invariants of the extended classical Galilei algebra

(1) $\Sigma_1 = u_t + \frac{1}{2}u_x^2 \qquad \Sigma_2 = u_x u_{xx} + u_{tx}$

$$\Sigma_3 = u_{tt}u_{xx} - u_{tx}^2 \qquad \Sigma_4 = u_{xx}. \tag{38}$$

(2) $\Sigma_1 = u^{-2}u_x^{-1}[u_x - 2uu_t] \qquad \Sigma_2 = (uu_x^{-1})^3 u_{xx}$

$\Sigma_3 = (uu_x)^{-3}[u^2u_x^3 - u^4u_tu_x^2 + u_tu_{xx} - u_xu_{tx}]$

$$\Sigma_4 = u^{-1}u_x^{-3}[u_x^2u_{tt} - 2u_tu_xu_{tx} + u_t^2u_{xx}]. \tag{39}$$

(3) $\Sigma_1 = u \qquad \Sigma_2 = x^2u_t \qquad \Sigma_3 = x^4u_{tt}$

$$\Sigma_4 = x^5[x(u_xu_{tt} - u_tu_{tx}) - 2u_t^2]. \tag{40}$$

$$(4)\ \Sigma_1 = (4u - x^2)u^{-2} \qquad \Sigma_2 = u^4u_t^{-4}(u_{tt}u_{xx} - u_{tx}^2). \tag{41}$$

Σ_3, Σ_4 are functionally-independent integrals of the first-order PDE

$$L \cdot F(\theta, \sigma, \rho) = 0$$

where

$$L = \theta\sqrt{4\theta - \alpha\theta^2}\partial_\theta - \sigma(\sqrt{4\theta - \alpha\theta^2} + 4\rho)\partial_\sigma + [\theta\sigma - 2\rho\sqrt{4\theta - \alpha\theta^2} - 2\sigma^{-1}(\beta\theta^{-4} + \rho^2)]\partial_\rho.$$

Variables θ, σ, ρ and parameters α, β in Σ_3, Σ_4 take the following form:

$$\theta = u \qquad \sigma = u_t^{-3}u_{tt} \qquad \rho = u_t^{-3}(u_xu_{tt} - u_tu_{tx}) \qquad \text{and } \alpha = \Sigma_1 \qquad \beta = \Sigma_2.$$

Note that we have chosen in (27) $\varphi(u) = 0$ and $\gamma(x) = \frac{x^2}{2}$.

Consequently, we have constructed the following classes of Galilei-invariant equations (1):

$$\Psi(J_1, J_2, J_3, J_4, J_5) = 0 \tag{42}$$

where $J_1, \ldots, J_5$ are given in (34)–(37), and

$$\overline{\Psi}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = 0 \tag{43}$$

where $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ are given in (38)–(41).

The family of invariant equations (42) includes the Burger's equation

$$u_t + uu_x + u_{xx} = 0$$

obtained by putting $J_1 + J_5 = 0$, where J_1, J_5 are given in (36), and the Monge–Amperé equation

$$u_{tt}u_{xx} - u_{tx}^2 = 0$$

obtained by putting $J_4 = 0$, where J_4 is give in (34) or (35) or (36).

The family of invariant equations (43) includes the modified Burger's equation

$$u_t + \tfrac{1}{2}u_x^2 + u_{xx} = 0$$

obtained by putting $\Sigma_1 + \Sigma_4 = 0$, where Σ_1, Σ_4 are given in (38), and the Monge–Amperé equation, obtained by putting $\Sigma_3 = 0$, where Σ_3 is given in (38).

5. Concluding remarks

We have shown that classification of equations invariant under some spacetime group involves as a first step the classification of possible realizations by LVFs of the corresponding Lie algebra. These realizations can be used for classification nth-order scalar Galilei-invariant PDEs with $n > 2$. Let us mention as an example the Korteweg–de Vries equation

$$u_t + u_{xxx} + uu_x = 0$$

which is invariant under classical Galilei algebra $AG_1(1, 1)$.

Since equations obtained admit by construction non-trivial Lie symmetry groups, one can apply the symmetry reduction technique to find their exact solutions. This is done via reduction of the two-dimensional PDEs (42), (43) to ordinary differential equations with the help of special substitutions (invariant solutions) [1].

These and other related problems are under investigation now and will be a topic of our future publications.

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